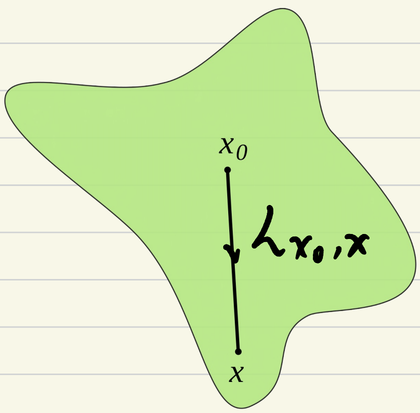
Tutorial 5

Q.1

Prove that any vector field F on a star-shaped region in $\mathbb{R}^n$ satisfying the "compatibility conditions" must be conservative.

Prerequisite:

Star-shaped region: A set $\Omega \subseteq \mathbb{R}^n$ is said to be star-shaped if $\exists x_0 \in \Omega$ s.t. $\forall x \in \Omega$, $\exists$ a straight line $L_{x_0,x} \subseteq \Omega$ joining x_0 & x.



Compatibility conditions:

A vector field $F: \Omega \to \mathbb{R}^n$ is said to satisfy the compatibility conditions if it is C^1 & $\forall\, 1 \le j, k \le n$, we have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial F_k}{\partial x_j}$$

<u>Remark</u>: The compatibility condition is trivial when $k=j$.

Here are the non-trivial compatibility conditions when $n=2,3$.

$n=2$: $\dfrac{\partial F_1}{\partial x_2} = \dfrac{\partial F_2}{\partial x_1}$

In general, there are $\frac{n(n-1)}{2}$ non-trivial compatibility conditions

$n=3$: $\dfrac{\partial F_1}{\partial x_2} = \dfrac{\partial F_2}{\partial x_1}$, $\dfrac{\partial F_1}{\partial x_3} = \dfrac{\partial F_3}{\partial x_1}$, $\dfrac{\partial F_2}{\partial x_3} = \dfrac{\partial F_3}{\partial x_2}$

Solution:

By lecture note 12, we know that if we can find a C^1 function $f:\Omega \to \mathbb{R}$ s.t. $\nabla f = F$, then F is conservative.

Define $f:\Omega \to \mathbb{R}$ by

$$f(x) = \int_{\{x_0, x\}} F \cdot d\vec{r}$$

$$= \int_0^1 F(xt + x_0(1-t)) \cdot (x - x_0)\, dt$$

We want to show that f is C^1, & $\nabla f = F$.

We first compute the partial derivatives of f.

$$\frac{\partial f}{\partial x_i}(x)$$

$$= \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$$

$$= \lim_{h\to 0} \int_0^1 \frac{\begin{pmatrix} F((x+he_i)t + x_0(1-t)) \cdot (x+he_i - x_0) \\ - F(xt + x_0(1-t)) \cdot (x - x_0) \end{pmatrix}}{h}\, dt$$

$$\overset{(?)}{=} \int_0^1 \lim_{h \to 0} \frac{\begin{pmatrix} F((x+he_i)t + x_0(1-t)) \cdot (x+he_i - x_0) \\ - F(xt + x_0(1-t)) \cdot (x - x_0) \end{pmatrix}}{h}\, dt$$

$$= \int_0^1 \frac{\partial}{\partial x_i}\big(F(xt + x_0(1-t)) \cdot (x - x_0)\big)\, dt \qquad -(*)$$

(?): to be verified later.

Note that

$$\frac{\partial}{\partial x_i}\left(F(xt + x_0(1-t))\cdot(x - x_0)\right)$$

$$= \frac{\partial}{\partial x_i}\sum_{j=1}^{n} F^j(xt + x_0(1-t))(x^j - x_0^j)$$

where upper script j stands for the j^{th} component

$$= \sum_{j=1}^{n}\frac{\partial}{\partial x_i}\left(F^j(xt+x_0(1-t))\right)(x^j - x_0^j) + F^j(xt + x_0(1-t))\frac{\partial}{\partial x_i}(x^j - x_0^j)$$

$$= \left[\sum_{j=1}^{n}\left(\frac{\partial F^j}{\partial x_i}(xt + x_0(1-t))\right)(t)(x^j - x_0^j)\right] + F^i(xt + x_0(1-t))$$

$$\frac{\partial}{\partial x_i}x^j = \delta_i^j = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i\end{cases}$$

$$= t\left[\sum_{j=1}^{n}\frac{\partial F^i}{\partial x^j}(xt + x_0(1-t))(x^j - x_0^j)\right] + F^i(xt + x_0(1-t))$$

compatibility condition

$$= t\left[\nabla F^i(xt + x_0(1-t))\right]\cdot(x - x_0) + F^i(xt + x(1-t))$$

$$= \frac{\partial}{\partial t}\left(t F^i(xt + x_0(1-t))\right)$$

$$(*) = \int_0^1 \frac{\partial}{\partial t}\left(t F^i(xt + x_0(1-t))\right)\, dt$$

$$= F^i(x) \text{ by FTC.}$$

$\therefore$ f is differentiable & $\nabla f = F$.

Since F is cont., f is C^1.

It is left to verify (?).

Fix $\varepsilon>0$, and define $G:[0,\varepsilon]\times[0,1]\to\mathbb{R}$ by

$$G(h,t)=\begin{cases}\dfrac{F((x+he_i)t+x_0(1-t))\cdot(x+he_i-x_0)-F(xt+x_0(1-t))\cdot(x-x_0)}{h}, & h\neq 0\\[2ex] \dfrac{\partial}{\partial x_i}\big(F(xt+x_0(1-t))\cdot(x-x_0)\big) & h=0\end{cases}$$

G is cont. on $[0,\varepsilon]\times[0,1]\Rightarrow$ uniformly cont. on $[0,\varepsilon]\times[0,1]$.

In particular, $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $\forall t\in[0,1]$, we have

$$|G(h,t)-G(0,t)|<\varepsilon \quad \text{as} \quad h<\delta.$$

$$G(0,t)-\varepsilon<G(h,t)<G(0,t)+\varepsilon$$

$$\int_0^1 G(0,t)\,dt-\varepsilon<\int_0^1 G(h,t)\,dt<\int_0^1 G(0,t)\,dt+\varepsilon$$

$$\Rightarrow \left|\int_0^1 G(h,t)\,dt-\int_0^1 G(0,t)\,dt\right|<\varepsilon \quad \text{as} \quad h<\delta$$

$$\therefore \lim_{h\to 0}\int_0^1 G(h,t)\,dt=\int_0^1 G(0,t)\,dt.$$

Q.2

Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field defined by

$$F(x,y,z) = (2x\cos y - 2z^3, 3+2ye^z - x^2\sin y, y^2e^z - 6xz^2)$$

(a) Show that F is conservative.

(b) Find all potential functions of F.

(c) Evaluate $\int_C F \cdot d\vec{r}$, where C is any differentiable curve starting from $(0,\pi,0)$ & ending at $(1,\pi,1)$

Solution:

(a) Since $\mathbb{R}^3$ is star-shaped, it suffices to show the compatibility conditions.

$$\frac{\partial F_1}{\partial y} = -2x\sin y, \quad \frac{\partial F_2}{\partial x} = -2x\sin y = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = -6z^2, \quad \frac{\partial F_3}{\partial x} = -6z^2 = \frac{\partial F_1}{\partial z}$$

$$\frac{\partial F_2}{\partial z} = 2ye^z, \quad \frac{\partial F_3}{\partial y} = 2ye^z = \frac{\partial F_2}{\partial z}$$

$\therefore$ F is conservative.

(b) A potential function $f: \mathbb{R}^3 \to \mathbb{R}$ satisfies $\nabla f = F$, i.e.

$$\frac{\partial f}{\partial x} = 2x\cos y - 2z^3 \qquad (1)$$

$$\frac{\partial f}{\partial y} = 3+2ye^z - x^2\sin y \qquad (2)$$

$$\frac{\partial f}{\partial z} = y^2e^z - 6xz^2 \qquad (3)$$

$(1) \Rightarrow f(x,y,z) = x^2\cos y - 2xz^3 + g_1(y,z)$

for some differentiable $g_1 : \mathbb{R}^2 \to \mathbb{R}$.

$(2) \Rightarrow -x^2\sin y + \frac{\partial g_1}{\partial y} = \frac{\partial f}{\partial y} = 3 + 2ye^z - x^2\sin y$

$$\therefore \frac{\partial g_1}{\partial y} = 3 + 2ye^z$$

$$g_1(y,z) = 3y + y^2e^z + g_2(z)$$

for some differentiable $g_2 : \mathbb{R} \to \mathbb{R}$.

$$f(x,y,z) = x^2\cos y - 2xz^3 + 3y + y^2e^z + g_2(z)$$

$(3) \Rightarrow \quad -6xz^2 + y^2e^z + \frac{dg_2}{dz} = \frac{\partial f}{\partial z} = y^2e^z - 6xz^2$

$$\frac{dg_2}{dz} = 0$$

$$\therefore g_2(z) = C = \text{constant}.$$

$\therefore$ The potential functions of F is

$f(x,y,z) = x^2\cos y - 2xz^3 + 3y + y^2e^z + C$, where $C \in \mathbb{R}$ is a constant.

(c) Choose $f(x,y,z) = x^2\cos y - 2xz^3 + 3y + y^2e^z$ as the potential function.

By the FTC for line integrals,

$$\int_C F \cdot d\vec{r} = f(1,\pi,1) - f(0,\pi,0)$$
$$= 1^2 \cdot \cos\pi - 2 + 3 + e - 0$$
$$= e$$